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Estimates and regularity results for the DiPerna-Lions flow

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Abstract. In this paper we derive new simple estimates for ordinary differential equations with Sobolev coefficients. These estimates not only allow to recover some old and recent results in a simple direct way, but they also have some new interesting corollaries.

1. Introduction

When $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a bounded smooth vector field, the *flow of* b is the smooth map $X : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{cases} \dfrac{dX}{dt}(t,x) = b\big(t, X(t,x)\big), & t \in [0,T], \\ X(0,x) = x. \end{cases} \tag{1}$$

Out of the smooth context (1) has been studied by several authors. In particular, the following is a common definition of generalized flow for vector fields which are merely integrable.

Definition 1.1 (Regular Lagrangian flow). Let $b \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^n; \mathbb{R}^n)$. We say that a map $X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a *regular Lagrangian flow* for the vector field b if

(i) for a.e. $x \in \mathbb{R}^n$ the map $t \mapsto X(t,x)$ is an absolutely continuous integral solution of $\dot\gamma(t) = b\big(t, \gamma(t)\big)$ for $t \in [0,T]$, with $\gamma(0) = x$;

(ii) there exists a constant L independent of t such that

$$\mathscr{L}^n\big(X(t,\cdot)^{-1}(A)\big) \leqq L\mathscr{L}^n(A) \quad \text{for every Borel set } A \subseteqq \mathbb{R}^n. \tag{2}$$

The constant L in (ii) will be called the *compressibility constant* of X.

Existence, uniqueness and stability of regular Lagrangian flows have been proved in [9] by DiPerna and Lions for Sobolev vector fields with bounded divergence. In a recent groundbreaking paper (see [1]) this result has been extended by Ambrosio to BV coefficients with bounded divergence.

The arguments of the DiPerna-Lions theory are quite indirect and they exploit (via the theory of characteristics) the connection between (1) and the Cauchy problem for the *transport equation*

$$\begin{cases} \partial_t u(t,x) + b(t,x) \cdot \nabla_x u(t,x) = 0, \\ u(0,\cdot) = \bar{u}. \end{cases} \tag{3}$$

Assuming that the divergence of b is in L^1 we can define bounded distributional solutions of (3) using the identity $b \cdot \nabla_x u = \nabla_x \cdot (bu) - u \nabla_x \cdot b$. Following DiPerna and Lions we say that a distributional solution $u \in L^\infty([0,T] \times \mathbb{R}^n)$ of (3) is a *renormalized solution* if

$$\begin{cases} \partial_t \big[\beta\big(u(t,x)\big)\big] + b(t,x) \cdot \nabla_x \big[\beta\big(u(t,x)\big)\big] = 0, \\ [\beta(u)](0,\cdot) = \beta(\bar{u}) \end{cases} \tag{4}$$

holds in the sense of distributions for every test function $\beta \in C^1(\mathbb{R};\mathbb{R})$. In their seminal paper DiPerna and Lions showed that, if the vector field b has Sobolev regularity with respect to the space variable, then every bounded solution is renormalized. Ambrosio [1] extended this result to BV vector fields with divergence in L^1. Under suitable compressibility assumptions (for instance $\nabla_x \cdot b \in L^\infty$), the renormalization property gives *uniqueness* and *stability* for (3) (the existence follows in a quite straightforward way from standard approximation procedures).

In turn, this uniqueness and stability property for (3) can be used to show existence, uniqueness and stability of regular Lagrangian flows (we refer to [9] for the original proofs and to [1] for a different derivation of the same conclusions).

In this paper we show how many of the ODE results of the DiPerna-Lions theory can be recovered from simple a priori estimates, directly in the Lagrangian formulation. Though our approach works under various relaxed hypotheses, namely controlled growth at infinity of the field b and L^p_{loc} and $L \log L$ assumptions on $D_x b$, for simplicity let us consider a vector field b in $W^{1,p} \cap L^\infty$, $p > 1$. Assuming the existence of a regular Lagrangian flow X, we give estimates of integral quantities depending on $X(t,x) - X(t,y)$. These estimates depend only on $\|b\|_{W^{1,p}} + \|b\|_\infty$ and the compressibility constant L of Definition 1.1(ii). Moreover, a similar estimate can be derived for the difference $X(t,x) - X'(t,x)$ of regular Lagrangian flows of different vector fields b and b', depending only on the compressibility constant of b and on $\|b\|_{W^{1,p}} + \|b\|_\infty + \|b'\|_\infty + \|b - b'\|_{L^1}$. As direct corollaries of our estimates we then derive:

(a) Existence, uniqueness, stability, and compactness of regular Lagrangian flows.

(b) Some mild regularity properties, like the approximate differentiability proved in [5], that we recover in a new quantitative fashion.

The regularity property in (b) has an effect on solutions to (3): we can prove that, for $b \in W^{1,p} \cap L^\infty$ with bounded divergence, solutions of (3) propagate the same mild regularity of the corresponding regular Lagrangian flow (we refer to Section 5 for the precise statements).

Our approach has been inspired by a recent result of Ambrosio, Lecumberry and Maniglia [5], proving the almost everywhere approximate differentiability of regular Lagrangian flows. Indeed, some of the quantities we estimate in this paper are taken directly from [5], whereas others are just suitable modifications. However, the way we derive our estimates is different: our analysis relies all on the Lagrangian formulation, whereas that of [5] relies on the Eulerian one.

Unfortunately we do not recover all the results of the theory of renormalized solutions. The main problem is that our estimates do not cover the case $Db \in L^1$. Actually, the extension to the case $Db \in L^1$ of our (or of similar) estimates would answer positively to the following conjecture of Bressan (see [6]):

Conjecture 1.2 (Bressan's compactness conjecture). *Let $b_k : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $k \in \mathbb{N}$, be smooth maps and denote by Φ_k the solutions of the* ODEs:

$$\begin{cases} \dfrac{d}{dt}\Phi_k(t,x) = b_k\big(t,\Phi_k(t,x)\big), \\ \Phi_k(0,x) = x. \end{cases} \tag{5}$$

Assume that $\|b_k\|_\infty + \|\nabla b_k\|_{L^1}$ is uniformly bounded and that the fluxes Φ_k are nearly incompressible, i.e. that

$$C^{-1} \leqq \det\big(\nabla_x \Phi_k(t,x)\big) \leqq C \quad \textit{for some constant } C > 0. \tag{6}$$

Then the sequence $\{\Phi_k\}$ is strongly precompact in L^1_{loc}.

At the present stage, the theory of renormalized solutions cannot be extended to cover this interesting case (we refer to [4] and to the survey article [8] for the results achieved so far in the framework of renormalized solutions). In another paper, [7], Bressan raised a second conjecture on mixing properties of flows of BV vector fields (see Conjecture 6.1 below), which can be considered as a quantitative version of Conjecture 1.2. In Section 6 we show how our estimates settle the $W^{1,p}$ $(p > 1)$ analog of Bressan's mixing conjecture.

In order to keep the presentation simple, in Section 2 we give the estimates and the various corollaries in the case $b \in W^{1,p} \cap L^\infty$ and in Section 3 we present the more general estimates and their consequences. We thank Herbert Koch for suggesting us that the Lipschitz estimates hold under the assumption $Db \in L \log L$ (see Remark 2.4 and the discussion at the beginning of Section 4). In Section 4 we show how to prove directly, via suitable a priori estimates, the compactness conclusion of Conjecture 1.2 when Db_k is bounded in $L \log L$. It has been pointed out to us independently by François Bouchut and by Pierre-Emmanuel Jabin that a more careful analysis allows to extend this approach when the sequence $\{Db_k\}$ is equi-integrable. In Section 5 we discuss the regularity results for transport equations mentioned above. Finally, in Section 6 we prove the $W^{1,p}$ analog of Bressan's mixing conjecture.

1.1. Notation and preliminaries. Constants will be denoted by c and $c_{a_1,\dots,a_q}$, where we understand that in the first case the constant is universal and in the latter it depends only on the quantities $a_1,\dots,a_q$. Therefore, during several computations, we will use the same symbol for constants which change from line to line. When A is a measurable subset of $\mathbb{R}^n$ we denote by $|A|$ or by $\mathscr{L}^n(A)$ its Lebesgue measure. When $f : \mathbb{R}^n \supset U \to V$ is continuous, we denote by $\mathrm{Lip}(f)$ the Lipschitz constant of f. When f is measurable we define

$$\mathrm{Lip}(f) := \min\{\mathrm{Lip}(g) \colon g \text{ is continuous and } g = f \text{ almost everywhere}\}.$$

When μ is a measure on Ω and $f : \Omega \to \Omega'$ a measurable map, $f_{\#}\mu$ will denote the push-forward of μ, i.e. the measure ν such that $\int \varphi \, d\nu = \int \varphi \circ f \, d\mu$ for every $\varphi \in C_c(\Omega')$.

2. A priori estimates for bounded vector fields and corollaries

In this section we show our estimates in the particular case of bounded vector fields. This estimate and its consequences are just particular cases of the more general theorems presented in the next sections. However, we decided to give independent proofs in this simplified setting in order to illustrate better the basic ideas of our analysis.

2.1. Estimate of an integral quantity and Lipschitz estimates.

Theorem 2.1. *Let b be a bounded vector field belonging to $L^1\big([0,T]; W^{1,p}(\mathbb{R}^n)\big)$ for some $p > 1$ and let X be a regular Lagrangian flow associated to b. Let L be the compressibility constant of X, as in Definition* 1.1(ii). *For every $p > 1$ define the following integral quantity*:

$$A_p(R,X) = \left[\int_{B_R(0)} \left(\sup_{0 \leqq t \leqq T} \sup_{0<r<2R} \fint_{B_r(x)} \log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1 \right) dy \right)^p dx \right]^{1/p}.$$

Then we have

$$A_p(R,X) \leqq C(R, L, \|D_x b\|_{L^1(L^p)}). \tag{7}$$

Remark 2.2. A small variant of the quantity $A_1(R,X)$ was first introduced in [5] and studied in an Eulerian setting in order to prove the approximate differentiability of regular Lagrangian flows. One basic observation of [5] is that a control of $A_1(R,X)$ implies the Lipschitz regularity of X outside of a set of small measure. This elementary Lipschitz estimate is shown in Proposition 2.3. The novelty of our point of view is that a direct Lagrangian approach allows to derive uniform estimates as in (7). These uniform estimates are then exploited in the next subsections to show existence, uniqueness, stability and regularity of the regular Lagrangian flow.

All the computations in the following proof can be justified using the definition of regular Lagrangian flow: the differentiation of the flow with respect to the time gives the vector field (computed along the flow itself), thanks to condition (i); condition (ii) implies that all the changes of variable we are performing just give an L in front of the integral.

During the proof, we will use some tools borrowed from the theory of maximal functions. We recall that, for a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$, the *local maximal function* is defined as

$$M_\lambda f(x) = \sup_{0<r<\lambda} \fint_{B_r(x)} |f(y)|\, dy.$$

For more details about the maximal function and for the statements of the lemmas we are going to use, we refer to Appendix A.

Proof of Theorem 2.1. For $0 \leqq t \leqq T$, $0 < r < 2R$ and $x \in B_R(0)$ define

$$Q(t,x,r) := \fint_{B_r(x)} \log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1\right) dy.$$

From Definition 1.1(i) it follows that for a.e. x and for every $r > 0$ the map $t \mapsto Q(t,x,r)$ is Lipschitz and

$$\text{(8)} \qquad \begin{aligned} \frac{dQ}{dt}(t,x,r) &\leqq \fint_{B_r(x)} \left|\frac{dX}{dt}(t,x) - \frac{dX}{dt}(t,y)\right| \big(|X(t,x) - X(t,y)| + r\big)^{-1} dy \\ &= \fint_{B_r(x)} \frac{\big|b\big(t, X(t,x)\big) - b\big(t, X(t,y)\big)\big|}{|X(t,x) - X(t,y)| + r}\, dy. \end{aligned}$$

We now set $\tilde{R} = 4R + 2T\|b\|_\infty$. Since we clearly have $|X(t,x) - X(t,y)| \leqq \tilde{R}$, applying Lemma A.3 we can estimate

$$\text{(9)} \qquad \begin{aligned} \frac{dQ}{dt}(t,x,r) &\leqq c_n \fint_{B_r(x)} \Big(M_{\tilde{R}} Db\big(t, X(t,x)\big) \\ &\qquad + M_{\tilde{R}} Db\big(t, X(t,y)\big)\Big) \frac{|X(t,x) - X(t,y)|}{|X(t,x) - X(t,y)| + r}\, dy \\ &\leqq c_n M_{\tilde{R}} Db\big(t, X(t,x)\big) + c_n \fint_{B_r(x)} M_{\tilde{R}} Db\big(t, X(t,y)\big)\, dy. \end{aligned}$$

Integrating with respect to the time, passing to the supremum for $0 < r < 2R$ and exchanging the supremums we obtain

$$\text{(10)} \qquad \begin{aligned} \sup_{0 \leqq t \leqq T} \sup_{0<r<2R} Q(t,x,r) &\leqq c + c_n \int_0^T M_{\tilde{R}} Db\big(t, X(t,x)\big)\, dt \\ &\qquad + c_n \int_0^T \sup_{0<r<2R} \fint_{B_r(x)} M_{\tilde{R}} Db\big(t, X(t,y)\big)\, dy\, dt. \end{aligned}$$

Taking the L^p norm over $B_R(0)$ we get

$$\text{(11)} \qquad A_p(R,X) \leqq c_{p,R} + c_n \left\| \int_0^T M_{\tilde{R}} Db\big(t, X(t,x)\big)\, dt \right\|_{L^p(B_R(0))}$$

$$\text{(12)} \qquad + c_n \left\| \int_0^T \sup_{0<r<2R} \fint_{B_r(x)} M_{\tilde{R}} Db\big(t, X(t,y)\big)\, dy\, dt \right\|_{L^p(B_R(0))}.$$

Recalling Definition 1.1(ii) and Lemma A.2, the integral in (11) can be estimated with

$$(13)\qquad c_n L^{1/p} \int_0^T \|M_{\tilde R} Db(t,x)\|_{L^p(B_{R+T\|b\|_\infty}(0))}\,dt \leqq c_{n,p} L^{1/p} \int_0^T \|Db(t,x)\|_{L^p(B_{R+\tilde R+T\|b\|_\infty}(0))}\,dt.$$

The integral in (12) can be estimated in a similar way with

$$\begin{aligned}
(14)\qquad & c_n \int_0^T \left\| \sup_{0<r<2R} \fint_{B_r(x)} \left[(M_{\tilde R} Db)\circ\big(t, X(t,\cdot)\big)\right](y)\,dy \right\|_{L^p(B_R(0))} dt \\
&= c_n \int_0^T \left\| M_{2R}\left[(M_{\tilde R} Db)\circ\big(t, X(t,\cdot)\big)\right](x)\right\|_{L^p(B_R(0))} dt \\
&\leqq c_{n,p} \int_0^T \left\| \left[(M_{\tilde R} Db)\circ\big(t, X(t,\cdot)\big)\right](x)\right\|_{L^p(B_{3R}(0))} dt \\
&= c_{n,p} \int_0^T \left\| (M_{\tilde R} Db)\circ\big(t, X(t,x)\big)\right\|_{L^p(B_{3R}(0))} dt \\
&\leqq c_{n,p} L^{1/p} \int_0^T \|M_{\tilde R} Db(t,x)\|_{L^p(B_{3R+T\|b\|_\infty}(0))}\,dt \\
&\leqq c_{n,p} L^{1/p} \int_0^T \|Db(t,x)\|_{L^p(B_{3R+T\|b\|_\infty+\tilde R}(0))}\,dt.
\end{aligned}$$

Combining (11), (12), (13) and (14), we obtain the desired estimate for $A_p(R,X)$. $\square$

We now show how the estimate of the integral quantity gives a quantitative Lipschitz estimate.

Proposition 2.3 (Lipschitz estimates). *Let $X : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ be a map. Then, for every $\varepsilon > 0$ and every $R > 0$, we can find a set $K \subset B_R(0)$ such that $|B_R(0)\backslash K| \leqq \varepsilon$ and for any $0 \leqq t \leqq T$ we have*

$$\mathrm{Lip}\big(X(t,\cdot)|_K\big) \leqq \exp \frac{c_n A_p(R,X)}{\varepsilon^{1/p}}.$$

Proof. Fix $\varepsilon > 0$ and $R > 0$. We can suppose that the quantity $A_p(R,X)$ is finite, otherwise the thesis is trivial; under this assumption, thanks to (34) we obtain a constant

$$M = M\big(\varepsilon, p, A_p(R,X)\big) = \frac{A_p(R,X)}{\varepsilon^{1/p}}$$

and a set $K \subset B_R(0)$ with $|B_R(0)\backslash K| \leqq \varepsilon$ and

$$\sup_{0 \leqq t \leqq T} \sup_{0<r<2R} \fint_{B_r(x)} \log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1\right) dy \leqq M \quad \forall x \in K.$$

This clearly means that

$$\fint_{B_r(x)} \log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1\right) dy \leqq M \quad \text{for every } x \in K,\ t \in [0,T] \text{ and } r \in \,]0, 2R[.$$

Now fix $x, y \in K$. Clearly $|x - y| < 2R$. Set $r = |x - y|$ and compute

$$\begin{aligned}
&\log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1\right) \\
&\quad = \fint_{B_r(x)\cap B_r(y)} \log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1\right) dz \\
&\quad \leqq \fint_{B_r(x)\cap B_r(y)} \log\left(\frac{|X(t,x) - X(t,z)|}{r} + 1\right) + \log\left(\frac{|X(t,y) - X(t,z)|}{r} + 1\right) dz \\
&\quad \leqq c_n \fint_{B_r(x)} \log\left(\frac{|X(t,x) - X(t,z)|}{r} + 1\right) dz + c_n \fint_{B_r(y)} \log\left(\frac{|X(t,y) - X(t,z)|}{r} + 1\right) dz \\
&\quad \leqq c_n M = \frac{c_n A_p(R,X)}{\varepsilon^{1/p}}.
\end{aligned}$$

This implies that

$$|X(t,x) - X(t,y)| \leqq \exp\left(\frac{c_n A_p(R,X)}{\varepsilon^{1/p}}\right)|x - y| \quad \text{for every } x, y \in K.$$

Therefore

$$\mathrm{Lip}\big(X(t,\cdot)|_K\big) \leqq \exp\frac{c_n A_p(R,X)}{\varepsilon^{1/p}}. \quad \square$$

Remark 2.4. The quantitative Lipschitz estimates also hold under the assumption $b \in L^1\big([0,T]; W^{1,1}(\mathbb{R}^n)\big) \cap L^\infty([0,T] \times \mathbb{R}^n)$ and $M_\lambda Db \in L^1\big([0,T]; L^1(\mathbb{R}^n)\big)$ for every $\lambda > 0$. To see this we define

$$\Phi(x) = \int_0^T M_{\tilde{R}} Db\big(t, X(t,x)\big)\, dt$$

and we go back to (10), which can be rewritten as

$$\sup_{0 \leqq t \leqq T} \sup_{0<r<2R} Q(t,x,r) \leqq c + c_n \Phi(x) + c_n M_{2R} \Phi(x).$$

For $\varepsilon < 1/(4c)$ we can estimate

$$
\begin{aligned}
&\left|\left\{x \in B_R(0) : c + c_n\Phi(x) + c_n M_{2R}\Phi(x) > \frac{1}{\varepsilon}\right\}\right| \\
&\qquad \leqq \left|\left\{x \in B_R(0) : c_n\Phi(x) > \frac{1}{4\varepsilon}\right\}\right| + \left|\left\{x \in B_R(0) : c_n M_{2R}\Phi(x) > \frac{1}{2\varepsilon}\right\}\right| \\
&\qquad \leqq \varepsilon c_n \int_{B_R(0)} \Phi(x)\,dx + \varepsilon c_n \int_{B_{3R}(0)} \Phi(x)\,dx \\
&\qquad \leqq \varepsilon c_n \int_0^T \int_{B_{3R}(0)} M_{\tilde{R}} Db\bigl(t, X(t,x)\bigr)\,dx\,dt \\
&\qquad \leqq \varepsilon c_n L \int_0^T \int_{B_{3R+T\|b\|_\infty}(0)} M_{\tilde{R}} Db(t,x)\,dx\,dt,
\end{aligned}
$$

where in the third line we applied the Chebyshev inequality and the weak estimate (33) and in the last line Definition 1.1(ii). This means that it is possible to find a set $K \subset B_R(0)$ with $|B_R(0)\backslash K| \leqq \varepsilon$ such that

$$
\fint_{B_r(x)} \log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1\right) dy \leqq \frac{c_n L}{\varepsilon} \int_0^T \int_{B_{3R+T\|b\|_\infty}(0)} M_{\tilde{R}} Db(t,x)\,dx\,dt
$$

for every $x \in K$, $t \in [0,T]$ and $r \in]0, 2R[$. Arguing as in the final part of the proof of Proposition 2.3 we obtain the Lipschitz estimate also in this case.

2.2. Existence, regularity and compactness. In this subsection we collect three direct corollaries of the estimates derived above, concerning approximate differentiability, existence and compactness of regular Lagrangian flows.

Corollary 2.5 (Approximate differentiability of the flow). *Let b be a bounded vector field belonging to $L^1\bigl([0,T]; W^{1,p}(\mathbb{R}^n)\bigr)$ for some $p > 1$, or belonging to $L^1\bigl([0,T]; W^{1,1}(\mathbb{R}^n)\bigr)$ and satisfying $M_\lambda Db \in L^1\bigl([0,T]; L^1(\mathbb{R}^n)\bigr)$ for every $\lambda > 0$, and let X be a regular Lagrangian flow associated to b. Then $X(t,\cdot)$ is approximately differentiable a.e. in $\mathbb{R}^n$, for every $t \in [0,T]$.*

Proof. The proof is an immediate consequence of the Lusin type approximation of the flow with Lipschitz maps given in Proposition 2.3 and Remark 2.4 and of Theorem B.1. $\square$

Corollary 2.6 (Compactness of the flow). *Let $\{b_h\}$ be a sequence of vector fields equi-bounded in $L^\infty([0,T] \times \mathbb{R}^n)$ and in $L^1\bigl([0,T]; W^{1,p}(\mathbb{R}^n)\bigr)$ for some $p > 1$. For each h, let X_h be a regular Lagrangian flow associated to b_h and let L_h be the compressibility constant of X_h, as in Definition 1.1(ii). Suppose that the sequence $\{L_h\}$ is equi-bounded. Then the sequence $\{X_h\}$ is strongly precompact in $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^n)$.*

Proof. Fix $\delta > 0$ and $R > 0$. Since $\{b_h\}$ is equi-bounded in $L^\infty([0,T] \times \mathbb{R}^n)$, we deduce that $\{X_h\}$ is equi-bounded in $L^\infty\bigl([0,T] \times B_R(0)\bigr)$: let $C_1(R)$ be an upper bound for these norms. Applying Proposition 2.3, for every h we find a Borel set $K_{h,\delta}$ such that $|B_R(0)\backslash K_{h,\delta}| \leqq \delta$ and

$$\operatorname{Lip}\big(X_h(t,\cdot)|_{K_{h,\delta}}\big) \leqq \exp\frac{c_n A_p(R,X_h)}{\delta^{1/p}} \quad \text{for every } t \in [0,T].$$

Recall first Theorem 2.1 implies that $A_p(R,X_h)$ is equi-bounded with respect to h, because of the assumptions of the corollary. Moreover, using Definition 1.1(i) and thanks again to the equi-boundedness of $\{b_h\}$ in $L^\infty([0,T]\times\mathbb{R}^n)$, we deduce that there exists a constant $C_2^\delta(R)$ such that

$$\operatorname{Lip}(X_h|_{[0,T]\times K_{h,\delta}}) \leqq C_2^\delta(R).$$

If we now set $B_{h,\delta} = [0,T]\times K_{h,\delta}$ and $M_\delta = \max\{C_1(R), C_2^\delta(R)\}$, we are in the position to apply Lemma C.1 with $\Omega = [0,T]\times B_R(0)$. Then the sequence $\{X_h\}$ is precompact in measure in $[0,T]\times B_R(0)$, and by equi-boundedness in L^∞ we deduce that it is also precompact in $L^1\big([0,T]\times B_R(0)\big)$. Using a standard diagonal argument it is possible to conclude that $\{X_h\}$ is locally precompact in $L^1([0,T]\times\mathbb{R}^n)$. □

Corollary 2.7 (Existence of the flow). *Let b be a bounded vector field belonging to $L^1\big([0,T];W^{1,p}(\mathbb{R}^n)\big)$ for some $p>1$ and such that $[\operatorname{div} b]^- \in L^1\big([0,T];L^\infty(\mathbb{R}^n)\big)$. Then there exists a regular Lagrangian flow associated to b.*

Proof. This is a simple consequence of the previous corollary. Choose a positive convolution kernel in $\mathbb{R}^n$ and regularize b by convolution. It is simple to check that the sequence of smooth vector fields $\{b_h\}$ we have constructed satisfies the equi-bounds of the previous corollary. Moreover, since every b_h is smooth, for every h there is a unique regular Lagrangian flow associated to b_h, with compressibility constant L_h given by

$$L_h = \exp\left(\int_0^T \|[\operatorname{div} b_h(t,\cdot)]^-\|_{L^\infty(\mathbb{R}^n)}\,dt\right). \tag{15}$$

Thanks to the positivity of the chosen convolution kernel, the sequence $\{L_h\}$ is equi-bounded, then we can apply Corollary 2.6. It is then easy to check that every limit point of $\{X_h\}$ in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^n)$ is a regular Lagrangian flow associated to b. □

Remark 2.8. An analogous existence result could be obtained removing the hypothesis on the divergence of b, and assuming that there is some approximation procedure such that we can regularize b with equi-bounds on the compressibility constants of the approximating flows. This remark also applies to Corollaries 3.7 and 4.3.

2.3. Stability estimates and uniqueness. In this subsection we show an estimate similar in spirit to that of Theorem 2.1, but comparing flows for different vector fields. A direct corollary of this estimate is the stability (and hence the uniqueness) of regular Lagrangian flows.

Theorem 2.9 (Stability of the flow). *Let b and $\tilde b$ be bounded vector fields belonging to $L^1\big([0,T];W^{1,p}(\mathbb{R}^n)\big)$ for some $p>1$. Let X and $\tilde X$ be regular Lagrangian flows associated* *to b and $\tilde{b}$ respectively and denote by L and $\tilde{L}$ the compressibility constants of the flows. Then, for every time $\tau \in [0, T]$, we have*

$$\|X(\tau,\cdot) - \tilde{X}(\tau,\cdot)\|_{L^1(B_r(0))} \leqq C|\log(\|b - \tilde{b}\|_{L^1([0,\tau]\times B_R(0))})|^{-1},$$

where $R = r + T\|b\|_\infty$ and the constant C only depends on τ, r, $\|b\|_\infty$, $\|\tilde{b}\|_\infty$, L, $\tilde{L}$, and $\|D_x b\|_{L^1(L^p)}$.

Proof. Set $\delta := \|b - \tilde{b}\|_{L^1([0,\tau]\times B_R(0))}$ and consider the function

$$g(t) := \int_{B_r(0)} \log\left(\frac{|X(t,x) - \tilde{X}(t,x)|}{\delta} + 1\right) dx.$$

Clearly $g(0) = 0$ and after some standard computations we get

$$\begin{aligned}
(16)\qquad g'(t) &\leqq \int_{B_r(0)} \left|\frac{dX(t,x)}{dt} - \frac{d\tilde{X}(t,x)}{dt}\right| \big(|X(t,x) - \tilde{X}(t,x)| + \delta\big)^{-1}\, dx \\
&= \int_{B_r(0)} \frac{\big|b\big(t, X(t,x)\big) - \tilde{b}\big(t, \tilde{X}(t,x)\big)\big|}{|X(t,x) - \tilde{X}(t,x)| + \delta}\, dx \\
&\leqq \frac{1}{\delta} \int_{B_r(0)} \big|b\big(t, \tilde{X}(t,x)\big) - \tilde{b}\big(t, \tilde{X}(t,x)\big)\big|\, dx \\
&\quad + \int_{B_r(0)} \frac{\big|b\big(t, X(t,x)\big) - b\big(t, \tilde{X}(t,x)\big)\big|}{|X(t,x) - \tilde{X}(t,x)| + \delta}\, dx.
\end{aligned}$$

We set $\tilde{R} = 2r + T(\|b\|_\infty + \|\tilde{b}\|_\infty)$ and we apply Lemma A.3 to estimate the last integral as follows:

$$\int_{B_r(0)} \frac{\big|b\big(t, X(t,x)\big) - b\big(t, \tilde{X}(t,x)\big)\big|}{|X(t,x) - \tilde{X}(t,x)| + \delta}\, dx \leqq c_n \int_{B_r(0)} M_{\tilde{R}} Db\big(t, X(t,x)\big) + M_{\tilde{R}} Db\big(t, \tilde{X}(t,x)\big)\, dx.$$

Inserting this estimate in (16), setting $\tilde{r} = r + T\max\{\|b\|_\infty, \|\tilde{b}\|_\infty\}$, changing variables in the integrals and using Lemma A.2 we get

$$\begin{aligned}
g'(t) &\leqq \frac{\tilde{L}}{\delta} \int_{B_{r+T\|\tilde{b}\|_\infty}(0)} |b(t,y) - \tilde{b}(t,y)|\, dy + (\tilde{L} + L) \int_{B_{\tilde{r}}(0)} M_{\tilde{R}} Db(t,y)\, dy \\
&\leqq \frac{\tilde{L}}{\delta} \int_{B_{r+T\|\tilde{b}\|_\infty}(0)} |b(t,y) - \tilde{b}(t,y)|\, dy + c_n \tilde{r}^{n-n/p} (\tilde{L} + L) \|M_{\tilde{R}} Db(t,\cdot)\|_{L^p} \\
&\leqq \frac{\tilde{L}}{\delta} \int_{B_{r+T\|\tilde{b}\|_\infty}(0)} |b(t,y) - \tilde{b}(t,y)|\, dy + c_{n,p} \tilde{r}^{n-n/p} (\tilde{L} + L) \|Db(t,\cdot)\|_{L^p}.
\end{aligned}$$

For any $\tau \in [0, T]$, integrating the last inequality between 0 and τ we get

$$g(\tau) = \int_{B_r(0)} \log\left(\frac{|X(\tau,x) - \tilde{X}(\tau,x)|}{\delta} + 1\right) dx \leqq C_1, \tag{17}$$

where the constant C_1 depends on τ, r, $\|b\|_\infty$, $\|\tilde{b}\|_\infty$, L, $\tilde{L}$, and $\|D_x b\|_{L^1(L^p)}$.

Next we fix a second parameter $\eta > 0$ to be chosen later. Using the Chebyshev inequality we find a measurable set $K \subset B_r(0)$ such that $|B_r(0) \backslash K| \leqq \eta$ and

$$\log\left(\frac{|X(\tau,x) - \tilde{X}(\tau,x)|}{\delta} + 1\right) \leqq \frac{C_1}{\eta} \quad \text{for } x \in K.$$

Therefore we can estimate

$$\begin{aligned}
\int_{B_r(0)} |X(\tau,x) - \tilde{X}(\tau,x)|\, dx \tag{18}\\
&\leqq \eta\big(\|X(\tau,\cdot)\|_{L^\infty(B_r(0))} + \|\tilde{X}(\tau,\cdot)\|_{L^\infty(B_r(0))}\big) + \int_K |X(\tau,x) - \tilde{X}(\tau,x)|\, dx\\
&\leqq \eta C_2 + c_n r^n \delta\big(\exp(C_1/\eta)\big) \leqq C_3\big(\eta + \delta \exp(C_1/\eta)\big),
\end{aligned}$$

with C_1, C_2 and C_3 which depend only on T, r, $\|b\|_\infty$, $\|\tilde{b}\|_\infty$, L, $\tilde{L}$, and $\|D_x b\|_{L^1(L^p)}$. Without loss of generality we can assume $\delta < 1$. Setting $\eta = 2C_1|\log\delta|^{-1} = 2C_1(-\log\delta)^{-1}$, we have $\exp(C_1/\eta) = \delta^{-1/2}$. Thus we conclude

$$\int_{B_r(0)} |X(\tau,x) - \tilde{X}(\tau,x)|\, dx \leqq C_3(2C_1|\log\delta|^{-1} + \delta^{1/2}) \leqq C|\log\delta|^{-1}, \tag{19}$$

where C depends only on τ, r, $\|b\|_\infty$, $\|\tilde{b}\|_\infty$, L, $\tilde{L}$, and $\|D_x b\|_{L^1(L^p)}$. This completes the proof. □

Corollary 2.10 (Uniqueness of the flow). *Let b be a bounded vector field belonging to $L^1\big([0,T]; W^{1,p}(\mathbb{R}^n)\big)$ for some $p > 1$. Then the regular Lagrangian flow associated to b, if it exists, is unique.*

Proof. It follows immediately from the stability proved in Theorem 2.9. □

Remark 2.11 (Stability with weak convergence in time). Theorem 2.9 allows to show the stability when the convergence of the vector fields is just weak with respect to the time. This setting is in fact very natural in view of the applications to the theory of fluid mechanics (see [9], Theorem II.7, and [11], in particular Theorem 2.5). In particular, under suitable bounds on the sequence $\{b_h\}$, the following form of weak convergence with respect to the time is sufficient to get the thesis:

$$\int_0^T b_h(t,x)\eta(t)\, dt \to \int_0^T b(t,x)\eta(t)\, dt \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ for every } \eta \in C_c^\infty(0,T).$$

Indeed, fix a parameter $\varepsilon > 0$ and regularize with respect to the spatial variables only using a standard convolution kernel ρ_ε. We can rewrite the difference $X_h(t,x) - X(t,x)$ as

$$X_h(t,x) - X(t,x) = \big(X_h(t,x) - X_h^\varepsilon(t,x)\big) + \big(X_h^\varepsilon(t,x) - X^\varepsilon(t,x)\big) + \big(X^\varepsilon(t,x) - X(t,x)\big),$$

where X^ε and X_h^ε are the flows relative to the regularized vector fields b^ε and b_h^ε respectively. Now, it is simple to check that

- the last term goes to zero with ε, by the classical stability theorem (the quantitative version is not needed at this point);

- the first term goes to zero with ε, uniformly with respect to h: this is due to the fact that the difference $b_h^\varepsilon - b_h$ goes to zero in $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^n)$ uniformly with respect to h, if we assume a uniform control in $W^{1,p}$ on the vector fields $\{b_h\}$, hence we can apply Theorem 2.9, and we get the desired convergence;

- the second term goes to zero for $h \to \infty$ when ε is kept fixed, because we are dealing with flows relative to vector fields which are smooth with respect to the space variable, uniformly in time, and weak convergence with respect to the time is enough to get the stability.

In order to conclude, we fix an arbitrary $\delta > 0$ and we first find $\varepsilon > 0$ such that the norm of the third term is smaller than δ and such that the norm of the first term is smaller than δ for every h. For this fixed ε, we find h such that the norm of the second term is smaller than δ. With this choice of h we have estimated the norm of $X_h(t,x) - X(t,x)$ with 3δ, hence we get the desired convergence.

Remark 2.12 (Another way to show compactness). If we apply Theorem 2.9 to the flows $X(t,x)$ and $\tilde{X}(t,x) = X(t,x+h) - h$ relative to the vector fields $b(t,x)$ and $\tilde{b}(t,x) = b(t,x+h)$, where $h \in \mathbb{R}^n$ is fixed, we get for every $\tau \in [0,T]$

$$\begin{aligned}\|X(\tau,\cdot) - X(\tau,\cdot+h) - h\|_{L^1(B_r(0))} &\leqq C\big|\log\big(\|b(t,x) - b(t,x+h)\|_{L^1([0,\tau]\times B_R(0))}\big)\big|^{-1} \\ &\leqq \frac{C}{|\log(h)|}.\end{aligned}$$

Hence we have a uniform control on the translations in the space, and we can deduce a compactness result applying the Riesz-Fréchet-Kolmogorov compactness criterion (Lemma C.2).

3. Estimates for more general vector fields and corollaries

In this section we extend the previous results to more general vector fields, in particular we drop the boundedness condition on b. More precisely, we will consider vector fields $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following regularity assumptions:

(R1) $b \in L^1\big([0,T]; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)\big)$ for some $p > 1$.

(R2) We can write

$$\frac{b(t,x)}{1+|x|} = \tilde{b}_1(t,x) + \tilde{b}_2(t,x)$$

with $\tilde{b}_1(t,x) \in L^1\big([0,T]; L^1(\mathbb{R}^n)\big)$ and $\tilde{b}_2(t,x) \in L^1\big([0,T]; L^\infty(\mathbb{R}^n)\big)$.

Since we are now considering vector fields which are no more bounded, we have to take care of the fact that the flow will be no more locally bounded in $\mathbb{R}^n$. However, we can give an estimate of the measure of the set of the initial data such that the corresponding trajectories exit from a fixed ball at some time.

Definition 3.1 (Sublevels). Fix $\lambda > 0$ and let $X : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a locally summable map. We set

$$G_\lambda := \{x \in \mathbb{R}^n : |X(t,x)| \leqq \lambda \ \forall t \in [0, T]\}. \tag{20}$$

Proposition 3.2 (Uniform estimate of the superlevels). *Let b be a vector field satisfying assumption* (R2) *and let X be a regular Lagrangian flow associated to b, with compressibility constant L. Then we have*

$$|B_R(0) \backslash G_\lambda| \leqq g(R, \lambda),$$

where the function g only depends on $\|\tilde{b}_1\|_{L^1(L^1)}$, $\|\tilde{b}_2\|_{L^1(L^\infty)}$ and L; moreover $g(R,\lambda) \downarrow 0$ for R fixed and $\lambda \uparrow +\infty$.

Proof. Let ϕ_t be the density of $X(t,\cdot)(\mathbf{1}_{B_R(0)}\mathscr{L}^n)$ with respect to $\mathscr{L}^n$ and notice that, by the definition of push-forward and by Definition 1.1(ii), we have $\|\phi_t\|_1 = \omega_n R^n$ and $\|\phi_t\|_\infty \leqq L$. Thanks to Definition 1.1(i) we can compute

$$\begin{aligned}
\int_{B_R(0)} \sup_{0 \leqq t \leqq T} \log\left(\frac{1 + |X(t,x)|}{1+R}\right) dx &\leqq \int_{B_R(0)} \int_0^T \frac{\left|\frac{dX}{dt}(t,x)\right|}{1 + |X(t,x)|}\, dt\, dx \\
&= \int_0^T \int_{B_R(0)} \frac{\left|b\big(t, X(t,x)\big)\right|}{1 + |X(t,x)|}\, dx\, dt \\
&\leqq \int_0^T \int_{\mathbb{R}^n} \frac{|b(t,x)|}{1+|x|} \phi_t \, dx\, dt.
\end{aligned}$$

Using the Hölder inequality, for every decomposition of $b(t,x)/(1+|x|)$ as in assumption (R2) we get

$$\int_{B_R(0)} \sup_{0 \leqq t \leqq T} \log\left(\frac{1 + |X(t,x)|}{1+R}\right) dx \leqq L\|\tilde{b}_1\|_{L^1(L^1)} + \omega_n R^n \|\tilde{b}_2\|_{L^1(L^\infty)}.$$

From this estimate we easily obtain

$$|B_R(0) \backslash G_\lambda| \leqq \left[\log\left(\frac{1+\lambda}{1+R}\right)\right]^{-1} \left(L\|\tilde{b}_1\|_{L^1(L^1)} + \omega_n R^n \|\tilde{b}_2\|_{L^1(L^\infty)}\right),$$

and the right-hand side clearly has the properties of the function $g(R,\lambda)$ stated in the proposition. $\square$

3.1. Estimate of an integral quantity and Lipschitz estimates. We start with the definition of an integral quantity which is a generalization of the quantity $A_p(R,X)$ of Theorem 2.1. In this new setting we will need a third variable (the truncation parameter λ), hence we set

$$A_p(R,\lambda,X) \tag{21}$$

$$:= \left[\int_{B_R(0)\cap G_\lambda} \left(\sup_{0\leqq t\leqq T} \sup_{0<r<2R} \fint_{B_r(x)\cap G_\lambda} \log\left(\frac{|X(t,x)-X(t,y)|}{r} + 1 \right) dy \right)^p dx \right]^{\frac{1}{p}}$$

where the set G_λ is the sublevel relative to the map X, defined as in Definition 3.1.

In the following proposition, we show a bound on the quantity $A_p(R,\lambda,X)$ which corresponds to the bound on $A_p(R,X)$ in Theorem 2.1.

Theorem 3.3. *Let b be a vector field satisfying assumptions* (R1) *and* (R2) *and let X be a regular Lagrangian flow associated to b, with compressibility constant L. Then we have*

$$A_p(R,\lambda,X) \leqq C(R,L,\|D_x b\|_{L^1([0,T],L^p(B_{3\lambda}(0)))}).$$

Proof. We start as in the proof of Theorem 2.1, obtaining the validity of inequality (8) for every $x \in G_\lambda$. Since $|X(t,x)-X(t,y)| \leqq 2\lambda$, applying Lemma A.3 we deduce

$$\frac{dQ}{dt}(t,x,r) \leqq c_n M_{2\lambda} Db\big(t,X(t,x)\big) + c_n \fint_{B_r(x)\cap G_\lambda} M_{2\lambda} Db\big(t,X(t,y)\big)\,dy.$$

Then, arguing exactly as in the proof of Theorem 2.1, we get the estimate

$$A_p(R,\lambda,X) \leqq c_{p,R} + c_n \left\| \int_0^T M_{2\lambda} Db\big(t,X(t,x)\big)\,dt \right\|_{L^p(B_R(0)\cap G_\lambda)} \tag{22}$$

$$+ c_n \left\| \int_0^T \sup_{0<r<2R} \fint_{B_r(x)\cap G_\lambda} M_{2\lambda} Db\big(t,X(t,y)\big)\,dy\,dt \right\|_{L^p(B_R(0)\cap G_\lambda)}. \tag{23}$$

Recalling Definition 1.1(ii) and Lemma A.2, the integral in (22) can be estimated with

$$c_n L^{1/p} \int_0^T \|M_{2\lambda} Db(t,x)\|_{L^p(B_\lambda(0))}\,dt \leqq c_{n,p} L^{1/p} \int_0^T \|Db(t,x)\|_{L^p(B_{3\lambda}(0))}\,dt.$$

Define the characteristic function $\mathbf{1}_A$ of a subset A of $\mathbb{R}^n$ as

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

The integral in (23) can be estimated in a similar way with

$$
\begin{aligned}
c_n \int_0^T & \left\| \sup_{0<r<2R} \fint_{B_r(x)\cap G_\lambda} \big[(M_{2\lambda}Db)\circ\big(t, X(t,\cdot)\big)\big](y)\,dy \right\|_{L^p(B_R(0)\cap G_\lambda)} dt \\
&\leqq c_n \int_0^T \left\| \sup_{0<r<2R} \fint_{B_r(x)} \big[(M_{2\lambda}Db)\circ\big(t, X(t,\cdot)\big)\big](y)\mathbf{1}_{G_\lambda}(y)\,dy \right\|_{L^p(B_R(0)\cap G_\lambda)} dt \\
&= c_n \int_0^T \big\| M_{2R}\big[(M_{2\lambda}Db)\circ\big(t, X(t,\cdot)\big)\mathbf{1}_{G_\lambda}(\cdot)\big](x) \big\|_{L^p(B_R(0)\cap G_\lambda)} dt \\
&\leqq c_{n,p} \int_0^T \big\| \big[(M_{2\lambda}Db)\circ\big(t, X(t,\cdot)\big)\mathbf{1}_{G_\lambda}(\cdot)\big](x) \big\|_{L^p(B_{3R}(0))} dt \\
&= c_{n,p} \int_0^T \big\| (M_{2\lambda}Db)\circ\big(t, X(t,x)\big) \big\|_{L^p(B_{3R}(0)\cap G_\lambda)} dt \\
&\leqq c_{n,p} L^{1/p} \int_0^T \| M_{2\lambda}Db(t,x) \|_{L^p(B_\lambda(0))} dt \\
&\leqq c_{n,p} L^{1/p} \int_0^T \| Db(t,x) \|_{L^p(B_{3\lambda}(0))} dt.
\end{aligned}
$$

Then we obtain the desired estimate for $A_p(R,\lambda,X)$. $\square$

Proposition 3.4 (Lipschitz estimates). *Let X and b be as in Theorem 3.3. Then, for every $\varepsilon > 0$ and every $R > 0$, we can find $\lambda > 0$ and a set $K \subset B_R(0)$ such that $|B_R(0)\backslash K| \leqq \varepsilon$ and for any $0 \leqq t \leqq T$ we have*

$$
\mathrm{Lip}\big(X(t,\cdot)|_K\big) \leqq \exp \frac{c_n A_p(R,\lambda,X)}{\varepsilon^{1/p}}.
$$

Proof. The proof is exactly the proof of Proposition 2.3, with some minor modifications due to the necessity of a truncation on the sublevels of the flow. This can be done as follows. For $\varepsilon > 0$ and $R > 0$ fixed, we apply Proposition 3.2 to get a λ large enough such that $|B_R(0)\backslash G_\lambda| \leqq \varepsilon/2$. Next, using equation (34) and the finiteness of $A_p(R,\lambda,X)$, we obtain a constant

$$
M = M\big(\varepsilon, p, A_p(R,\lambda,X)\big) = \frac{A_p(R,\lambda,X)}{(\varepsilon/2)^{1/p}}
$$

and a set $K \subset B_R(0)\cap G_\lambda$ with $\big|\big(B_R(0)\cap G_\lambda\big)\backslash K\big| \leqq \varepsilon/2$ and

$$
\sup_{0\leqq t\leqq T} \sup_{0<r<2R} \fint_{B_r(x)\cap G_\lambda} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right) dy \leqq M \quad \forall x \in K.
$$

Hence the set K satisfies $|B_R(0)\backslash K| \leqq \varepsilon$ and

$$
\fint_{B_r(x)\cap G_\lambda} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right) dy \leqq M \quad \forall x \in K,\ \forall t \in [0,T],\ \forall r \in]0,2R[.
$$

The proof can be concluded as the proof of Proposition 2.3, where now the integrals are performed on the sublevels G_λ. □

3.2. Existence, regularity and compactness.

Corollary 3.5 (Approximate differentiability of the flow). *Let b be a vector field satisfying assumptions* (R1) *and* (R2) *and let X be a regular Lagrangian flow associated to b. Then $X(t,\cdot)$ is approximately differentiable a.e. in $\mathbb{R}^n$, for every $t \in [0,T]$.*

Proof. The proof is an immediate consequence of the Lusin type approximation of the flow with Lipschitz maps given in Proposition 3.4 and of Theorem B.1. □

Corollary 3.6 (Compactness of the flow). *Let $\{b_h\}$ be a sequence of vector fields satisfying assumptions* (R1) *and* (R2)*. For every h, let X_h be a regular Lagrangian flow associated to b_h and let L_h be the compressibility constant associated to X_h, as in Definition* 1.1(ii)*. Suppose that for every $R > 0$ the uniform estimate*

$$\|D_x b_h\|_{L^1([0,T];L^p(B_R(0)))} + \|\tilde{b}_{h,1}\|_{L^1(L^1)} + \|\tilde{b}_{h,2}\|_{L^1(L^\infty)} + L_h \leqq C(R) < \infty \tag{24}$$

is satisfied, for some decomposition $b_h/(1+|x|) = \tilde{b}_{h,1} + \tilde{b}_{h,2}$ as in assumption (R2)*. Then the sequence $\{X_h\}$ is locally precompact in measure in $[0,T] \times \mathbb{R}^n$.*

Proof. The proof is essentially identical to the proof of Corollary 2.6. Fix $R > 0$ and $\delta > 0$. Applying Proposition 3.2 and thanks to the uniform bound given by (24), we first find $\lambda > 0$ big enough such that

$$|B_R(0) \backslash G_\lambda^h| \leqq \delta/3,$$

with G_λ^h as in Definition 3.1. Thanks again to (24), we can apply Theorem 3.3 to deduce that the quantities $A_p(R,\lambda,X_h)$ are uniformly bounded with respect to h. Now we apply Proposition 3.4 with $\varepsilon = \delta/3$ to find, for every h, a measurable set $K_h \subset B_R(0) \cap G_\lambda^h$ such that

$$\big|\big(B_R(0) \cap G_\lambda^h\big) \backslash K_h\big| \leqq \delta/3$$

and

$$\mathrm{Lip}\big(X_h(t,\cdot)|_{K_h}\big) \text{ is uniformly bounded w.r.t. } h.$$

Now we are going to show a similar Lipschitz estimate with respect to the time. Since the maps

$$[0,T] \times K_h \ni (t,x) \mapsto b_h\big(t, X_h(t,x)\big)$$

are uniformly bounded in $L^1([0,T] \times K_h)$ (this is easily deduced recalling assumption (R2), the bound (24) and the fact that $K_h \subset B_R(0)$), for every h, applying the Chebyshev inequality, we can find a measurable set $H_h \subset [0,T] \times K_h$ such that

$$|([0,T]\times K_h)\backslash H_h| \leqq \delta/3$$

and

$$\big\|b_h\big(t, X_h(t,x)\big)\big\|_{L^\infty(H_h)} \leqq C/\delta,$$

where the constant C only depends on the constant $C(R)$ given by (24). Then we deduce that

$$\left\|\frac{dX_h}{dt}(t,x)\right\|_{L^\infty(H_h)} \text{ is uniformly bounded w.r.t. } h.$$

Hence we have found, for every h, a measurable set $H_h \subset [0,T]\times B_R(0)$ such that

$$\big|\big([0,T]\times B_R(0)\big)\backslash H_h\big| \leqq \delta$$

and

$$\|X_h\|_{L^\infty(H_h)} + \mathrm{Lip}_{t,x}(X_h|_{H_h}) \text{ uniformly bounded w.r.t. } h.$$

Then we apply Lemma C.1 to obtain that the sequence $\{X_h\}$ is precompact in measure in $[0,T]\times B_R(0)$. A standard diagonal argument gives the local precompactness in measure of the sequence in the whole $[0,T]\times\mathbb{R}^n$. $\square$

Corollary 3.7 (Existence of the flow). *Let b be a vector field satisfying assumptions* (R1) *and* (R2) *and such that $[\operatorname{div} b]^- \in L^1\big([0,T]; L^\infty(\mathbb{R}^n)\big)$. Then there exists a regular Lagrangian flow associated to b.*

Proof. It is sufficient to regularize b with a positive convolution kernel in $\mathbb{R}^n$ and apply Corollary 3.6. It is simple to check that the regularized vector fields satisfy the equi-bounds needed for the compactness result. $\square$

3.3. Stability estimates and uniqueness.

Theorem 3.8 (Stability estimate). *Let b and $\tilde b$ be vector fields satisfying assumptions* (R1) *and* (R2). *Let X and $\tilde X$ be regular Lagrangian flows associated to b and $\tilde b$ respectively and denote by L and $\tilde L$ the compressibility constants of the flows. Then for every $\lambda > 1$ and every $\tau \in [0,T]$ the following estimate holds*:

$$\int_{B_r(0)} 1 \wedge |X(\tau,x) - \tilde X(\tau,x)|\,dx \leqq \frac{C}{\log(\lambda)} + C_\lambda \|b - \tilde b\|_{L^1([0,\tau]\times B_\lambda(0))}, \tag{25}$$

where the constant C only depends on L, $\tilde L$ and on the $L^1(L^1) + L^1(L^\infty)$ norm of some decomposition of b and $\tilde b$ as in assumption (R2), *while the constant C_λ depends on λ, r, L, $\tilde L$ and $\|Db\|_{L^1([0,\tau];L^p(B_{3\lambda}(0)))}$.*

Proof. For any given $\lambda > 1$ define the sets G_λ and $\tilde G_\lambda$, relatively to X and $\tilde X$, as in (20). Set

$$\delta = \delta(\lambda) := \|b - \tilde{b}\|_{L^1([0,\tau]\times B_\lambda(0))}.$$

Define

$$g(t) := \int_{B_r(0)\cap G_\lambda\cap\tilde{G}_\lambda} \log\left(\frac{|X(t,x) - \tilde{X}(t,x)|}{\delta} + 1\right) dx.$$

Clearly we have $g(0) = 0$ and we can estimate

$$\begin{aligned}
g'(t) &\leqq \int_{B_r(0)\cap G_\lambda\cap\tilde{G}_\lambda} \frac{\left|b\big(t,X(t,x)\big) - \tilde{b}\big(t,\tilde{X}(t,x)\big)\right|}{|X(t,x)-\tilde{X}(t,x)|+\delta}\, dx \\
&\leqq \int_{B_r(0)\cap G_\lambda\cap\tilde{G}_\lambda} \frac{\left|b\big(t,\tilde{X}(t,x)\big) - \tilde{b}\big(t,\tilde{X}(t,x)\big)\right|}{|X(t,x)-\tilde{X}(t,x)|+\delta} + \frac{\left|b\big(t,X(t,x)\big) - b\big(t,\tilde{X}(t,x)\big)\right|}{|X(t,x)-\tilde{X}(t,x)|+\delta}\, dx \\
&\leqq \int_{B_r(0)\cap G_\lambda\cap\tilde{G}_\lambda} \frac{1}{\delta}\left|b\big(t,\tilde{X}(t,x)\big) - \tilde{b}\big(t,\tilde{X}(t,x)\big)\right| + \frac{\left|b\big(t,X(t,x)\big) - b\big(t,\tilde{X}(t,x)\big)\right|}{|X(t,x)-\tilde{X}(t,x)|}\, dx \\
&\leqq \frac{1}{\delta} \int_{B_r(0)\cap G_\lambda\cap\tilde{G}_\lambda} \left|b\big(t,\tilde{X}(t,x)\big) - \tilde{b}\big(t,\tilde{X}(t,x)\big)\right| dx \\
&\quad + c_n \int_{B_r(0)\cap G_\lambda\cap\tilde{G}_\lambda} \Big(M_{2\lambda}Db\big(t,X(t,x)\big) + M_{2\lambda}Db\big(t,\tilde{X}(t,x)\big)\Big)\, dx \\
&\leqq \frac{\tilde{L}}{\delta}\int_{B_\lambda(0)} |b(t,x) - \tilde{b}(t,x)|\, dx + c_n(L+\tilde{L}) \int_{B_\lambda(0)} M_{2\lambda}Db(t,x)\, dx \\
&\leqq \frac{\tilde{L}}{\delta}\int_{B_\lambda(0)} |b(t,x) - \tilde{b}(t,x)|\, dx + c_{n,p}(L+\tilde{L})\lambda^{n-n/p}\|Db(t,\cdot)\|_{L^p(B_{3\lambda}(0))}.
\end{aligned}$$

Integrating with respect to t between 0 and τ we obtain

$$\begin{aligned}
g(\tau) &= \int_{B_r(0)\cap G_\lambda\cap\tilde{G}_\lambda} \log\left(\frac{|X(\tau,x) - \tilde{X}(\tau,x)|}{\delta} + 1\right) dx \\
&\leqq \tilde{L} + c_{n,p}(L+\tilde{L})\lambda^{n-n/p}\|Db\|_{L^1([0,\tau];\,L^p(B_{3\lambda}(0)))} = C_\lambda,
\end{aligned}$$

where the constant C_λ depends on λ but also on the other parameters relative to b and $\tilde{b}$. Now fix a value $\eta > 0$ which will be specified later. We can find a measurable set $K \subset B_r(0)\cap G_\lambda\cap\tilde{G}_\lambda$ such that $\left|\big(B_r(0)\cap G_\lambda\cap\tilde{G}_\lambda\big)\backslash K\right| < \eta$ and

$$\log\left(\frac{|X(\tau,x) - \tilde{X}(\tau,x)|}{\delta} + 1\right) \leqq \frac{C_\lambda}{\eta} \quad \forall x \in K.$$

Then we deduce that

$$\int_{B_r(0)} 1 \wedge |X(\tau,x) - \tilde{X}(\tau,x)|\,dx$$

$$\leqq |B_r(0)\setminus(G_\lambda \cap \tilde{G}_\lambda)| + \big|\big(B_r(0) \cap G_\lambda \cap \tilde{G}_\lambda\big)\setminus K\big| + \int_K |X(t,x) - \tilde{X}(t,x)|\,dx$$

$$\leqq \frac{C}{\log(\lambda)} + \eta + C\delta \exp(C_\lambda/\eta) \leqq \frac{C}{\log(\lambda)} + C_\lambda \|b - \tilde{b}\|_{L^1([0,\tau]\times B_\lambda(0))},$$

choosing $\eta = 1/\log(\lambda)$ in the last line. □

Corollary 3.9 (Stability of the flow). *Let $\{b_h\}$ be a sequence of vector fields satisfying assumptions* (R1) *and* (R2), *converging in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^n)$ to a vector field b which satisfies assumptions* (R1) *and* (R2). *Denote by X and X_h the regular Lagrangian flows associated to b and b_h respectively, and denote by L and L_h the compressibility constants of the flows. Suppose that, for some decomposition $b_h/(1+|x|) = \tilde{b}_{h,1} + \tilde{b}_{h,2}$ as in assumption* (R2), *we have*

$$\|\tilde{b}_{h,1}\|_{L^1(L^1)} + \|\tilde{b}_{h,2}\|_{L^1(L^\infty)} \text{ equi-bounded in } h$$

and that the sequence $\{L_h\}$ is equi-bounded. Then the sequence $\{X_h\}$ converges to X locally in measure in $[0,T]\times\mathbb{R}^n$.

Proof. Notice that, under the hypothesis of this corollary, the constants $C^{h,\tau}$ and $C_\lambda^{h,\tau}$ in (25) can be chosen uniformly with respect to $\tau \in [0,T]$ and $h \in \mathbb{N}$. Hence we find universal constants C and C_λ, depending only on the assumed equi-bounds, such that

$$\int_{B_r(0)} 1 \wedge |X(\tau,x) - X_h(\tau,x)|\,dx \leqq \frac{C^{h,\tau}}{\log(\lambda)} + C_\lambda^{h,\tau} \|b - b_h\|_{L^1([0,\tau]\times B_\lambda(0))} \tag{26}$$

$$\leqq \frac{C}{\log(\lambda)} + C_\lambda \|b - b_h\|_{L^1([0,T]\times B_\lambda(0))}.$$

Now fix $\varepsilon > 0$. We first choose λ big enough such that

$$\frac{C}{\log(\lambda)} \leqq \frac{\varepsilon}{2},$$

where C is the first constant in (26). Since now λ is fixed, we find N such that for every $h \geqq N$ we have

$$\|b - b_h\|_{L^1([0,T]\times B_\lambda(0))} \leqq \frac{\varepsilon}{2C_\lambda},$$

thanks to the convergence of the sequence $\{b_h\}$ to b in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^n)$. Notice that N depends on λ and on the equi-bounds, but in turn λ only depends on ε and on the equi-bounds. Hence we get

$$\int_{B_r(0)} 1 \wedge |X(\tau,x) - X_h(\tau,x)|\,dx \leqq \varepsilon \quad \text{for every } h \geqq N = N(\varepsilon).$$

This means that $\{X_h(\tau,\cdot)\}$ converges to $X(\tau,\cdot)$ locally in measure in $\mathbb{R}^n$, uniformly with respect to $\tau \in [0,T]$. In particular we get the thesis. $\square$

Corollary 3.10 (Uniqueness of the flow). *Let b be a vector field satisfying assumptions* (R1) *and* (R2). *Then the regular Lagrangian flow associated to b, if it exists, is unique.*

Proof. It follows immediately from Corollary 3.9. $\square$

4. A direct proof of compactness

In this section we propose an alternative proof of the compactness result of Theorem 2.6, which works under an assumption of summability of the maximal function of Db. The strategy of this proof is slightly different from the previous one: we are not going to use the Lipschitz estimates of Proposition 2.3 and Remark 2.4, but instead we prove an estimate of an integral quantity which turns out to be sufficient to get compactness, via the Riesz-Fréchet-Kolmogorov compactness criterion.

We will assume the following regularity assumption on the vector field:

(R3) For every $\lambda > 0$ we have $M_\lambda Db \in L^1\big([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^n)\big)$.

Notice that, by Lemma A.2, this assumption is equivalent to the condition

$$\int_0^T \int_{B_\rho(0)} |D_x b(t,x)| \log\big(2 + |D_x b(t,x)|\big)\, dx\, dt < \infty \quad \text{for every } \rho > 0.$$

This means that $D_x b \in L^1\big([0,T]; L\log L_{\mathrm{loc}}(\mathbb{R}^n)\big)$, i.e. a slightly stronger bound than $D_x b \in L^1\big([0,T], L^1_{\mathrm{loc}}(\mathbb{R}^n)\big)$.

We define a new integral quantity, which corresponds to the one defined in Theorem 2.1 for $p = 1$, but without the supremum with respect to r. For $R > 0$ and $0 < r < R/2$ fixed we set

$$a(r,R,X) = \int_{B_R(0)} \sup_{0 \leqq t \leqq T} \fint_{B_r(x)} \log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1\right) dy\, dx.$$

We first give a quantitative estimate for the quantity $a(r,R,X)$, similar to the one for $A_p(R,X)$.

Theorem 4.1. *Let b be a bounded vector field satisfying assumption* (R3) *and let X be a regular Lagrangian flow associated to b, with compressibility constant L. Then we have*

$$a(r,R,X) \leqq C\big(R, L, \|M_{\tilde R} D_x b\|_{L^1([0,T];L^1(B_{\tilde R}(0)))}\big),$$

where $\tilde R = 3R/2 + 2T\|b\|_\infty$.

Proof. We start as in the proof of Theorem 2.1, obtaining inequality (9) (but this time it is sufficient to set $\tilde{R} = 3R/2 + 2T\|b\|_\infty$). Integrating with respect to the time and then with respect to x over $B_R(0)$, we obtain

$$a(r, R, X) \leqq c_R + c_n \int_{B_R(0)} \int_0^T M_{\tilde{R}} Db\big(t, X(t, x)\big)\, dt\, dx$$

$$+ c_n \int_{B_R(0)} \int_0^T \fint_{B_r(x)} M_{\tilde{R}} Db\big(t, X(t, y)\big)\, dy\, dt\, dx.$$

As in the previous computations, the first integral can be estimated with

$$c_n L \|M_{\tilde{R}} Db\|_{L^1([0, T]; L^1(B_{R+T\|b\|_\infty}(0)))},$$

but this time we cannot bound the norm of the maximal function with the norm of the derivative. To estimate the last integral we compute

$$c_n \int_{B_R(0)} \int_0^T \fint_{B_r(x)} M_{\tilde{R}} Db\big(t, X(t, y)\big)\, dy\, dt\, dx$$

$$= c_n \int_{B_R(0)} \int_0^T \fint_{B_r(0)} M_{\tilde{R}} Db\big(t, X(t, x+z)\big)\, dz\, dt\, dx$$

$$\leqq c_n \fint_{B_r(0)} \int_0^T \int_{B_R(0)} M_{\tilde{R}} Db\big(t, X(t, x+z)\big)\, dx\, dt\, dz$$

$$\leqq c_n \fint_{B_r(0)} \int_0^T L \int_{B_{3R/2+T\|b\|_\infty}(0)} M_{\tilde{R}} Db(t, w)\, dw\, dt\, dx$$

$$= c_n L \|M_{\tilde{R}} Db\|_{L^1([0, T]; L^1(B_{3R/2+T\|b\|_\infty}(0)))}.$$

Hence the thesis follows, by definition of $\tilde{R}$. □

Next, we show how this estimate implies compactness for the flow.

Corollary 4.2 (Compactness of the flow). *Let $\{b_h\}$ be a sequence of vector fields equi-bounded in $L^\infty([0, T] \times \mathbb{R}^n)$ and suppose that the sequence $\{M_\lambda Db_h\}$ is equi-bounded in $L^1\big([0, T]; L^1_{\text{loc}}(\mathbb{R}^n)\big)$ for every $\lambda > 0$. For each h, let X_h be a regular Lagrangian flow associated to b_h and let L_h be the compressibility constant associated to X_h, as in Definition* 1.1(ii). *Suppose that the sequence $\{L_h\}$ is equi-bounded. Then the sequence $\{X_h\}$ is strongly precompact in $L^1_{\text{loc}}([0, T] \times \mathbb{R}^n)$.*

Proof. We apply Theorem 4.1 to obtain that, under the assumptions of the corollary, the quantities $a(r, R, X_h)$ are uniformly bounded with respect to h. Now observe that, for $0 \leqq z \leqq \tilde{R}$ (with $\tilde{R} = 3R/2 + 2T\|b\|_\infty$ as in Theorem 4.1), thanks to the concavity of the logarithm we have

$$\log\left(\frac{z}{r}+1\right) \geqq \frac{\log\left(\frac{\tilde{R}}{r}+1\right)}{\tilde{R}} z.$$

Since $|X_h(t,x) - X_h(t,y)| \leqq \tilde{R}$ this implies that

$$\int_{B_R(0)} \sup_{0 \leqq t \leqq T} \fint_{B_r(x)} |X_h(t,x) - X_n(t,y)|\, dy\, dx$$

$$\leqq \frac{\tilde{R}}{\log\left(\frac{\tilde{R}}{r}+1\right)} C(R, L_h, \|M_{\tilde{R}} Db_h\|_{L^1([0,T];L^1(B_{\tilde{R}}(0)))}) \leqq g(r),$$

where the function $g(r)$ does not depend on h and satisfies $g(r) \downarrow 0$ for $r \downarrow 0$. Changing the integration order this implies

$$\fint_{B_r(0)} \int_{B_R(0)} |X_h(t,x) - X_h(t,x+z)|\, dx\, dz \leqq g(r),$$

uniformly with respect to t and h.

Now notice the following elementary fact. There exists a dimensional constant $\alpha_n > 0$ with the following property: if $A \subset B_1(0)$ is a measurable set with $|B_1(0) \backslash A| \leqq \alpha_n$, then $A + A \supset B_{1/2}(0)$. Indeed, if the thesis were false, we could find $x \in B_{1/2}(0)$ such that $x \notin A + A$. This would imply in particular that $x \notin \big(A \cap B_{1/2}(0)\big) + \big(A \cap B_{1/2}(0)\big)$, so that

$$\big[x - \big(A \cap B_{1/2}(0)\big)\big] \cap [A \cap B_{1/2}(0)] = \emptyset. \tag{27}$$

Now notice that there exists a dimensional constant γ_n such that

$$\big|B_{1/2}(0) \cap \big(x - B_{1/2}(0)\big)\big| \geqq \gamma_n,$$

since we are supposing $x \in B_{1/2}(0)$. But since $|B_1(0) \backslash A| \leqq \alpha_n$, we also have

$$\big|B_{1/2}(0) \backslash \big(A \cap B_{1/2}(0)\big)\big| \leqq \alpha_n$$

and

$$\big|\big(x - B_{1/2}(0)\big) \backslash \big(x - \big(A \cap B_{1/2}(0)\big)\big)\big| = \big|B_{1/2}(0) \backslash \big(A \cap B_{1/2}(0)\big)\big| \leqq \alpha_n.$$

But this is clearly in contradiction with (27) if we choose $\alpha_n < \gamma_n/2$.

Then fix α_n as above and apply the Chebyshev inequality for every h to obtain, for every $0 < r < R/2$, a measurable set $K_{r,h} \subset B_r(0)$ with $|B_r(0) \backslash K_{r,h}| \leqq \alpha_n |B_r(0)|$ and

$$\int_{B_R(0)} |X_h(t,x+z) - X_h(t,x)|\, dx \leqq \frac{g(r)}{\alpha_n} \quad \text{for every } z \in K_{r,h}.$$

For such a set $K_{r,h}$, thanks to the previous remark, we have that $K_{r,h} + K_{r,h} \supset B_{r/2}(0)$. Now let $v \in B_{r/2}(0)$ be arbitrary. For every h we can write $v = z_{1,h} + z_{2,h}$ with $z_{1,h}, z_{2,h} \in K_{r,h}$. We can estimate the increment in the spatial directions as follows:

$$
\begin{aligned}
&\int_{B_{R/2}(0)} |X_h(t, x + v) - X_h(t, x)|\, dx \\
&\quad = \int_{B_{R/2}(0)} |X_h(t, x + z_{1,h} + z_{2,h}) - X_h(t, x)|\, dx \\
&\quad \leqq \int_{B_{R/2}(0)} |X_h(t, x + z_{1,h} + z_{2,h}) - X_h(t, x + z_{1,h})| + |X_h(t, x + z_{1,h}) - X_h(t, x)|\, dx \\
&\quad \leqq \int_{B_R(0)} |X_h(t, y + z_{2,h}) - X_h(t, y)|\, dy + \int_{B_R(0)} |X_h(t, x + z_{1,h}) - X_h(t, x)|\, dx \leqq \frac{2g(r)}{\alpha_n}.
\end{aligned}
$$

Now notice that, by Definition 1.1(i), for a.e. $x \in \mathbb{R}^n$ we have

$$
\frac{dX_h}{dt}(t, x) = b_h\big(t, X_h(t, x)\big) \quad \text{for every } t \in [0, T].
$$

Then we can estimate the increment in the time direction in the following way:

$$
\begin{aligned}
|X_h(t + \tau, x) - X_h(t, x)| &\leqq \int_0^\tau \left| \frac{dX_h}{dt}(t + s, x) \right| ds \\
&= \int_0^\tau \big| b_h\big(t + s, X_h(t + s, x)\big)\big|\, ds \leqq \tau \|b_h\|_\infty.
\end{aligned}
$$

Combining these two informations, for $(t_0, t_1) \subset\subset [0, T]$, $R > 0$, $v \in B_{r/2}(0)$ and $\tau > 0$ sufficently small we can estimate

$$
\begin{aligned}
&\int_{t_0}^{t_1} \int_{B_{R/2}(0)} |X_h(t + \tau, x + v) - X_h(t, x)|\, dx\, dt \\
&\quad \leqq \int_{t_0}^{t_1} \int_{B_{R/2}(0)} |X_h(t + \tau, x + v) - X_h(t + \tau, x)| + |X_h(t + \tau, x) - X_h(t, x)|\, dx\, dt \\
&\quad \leqq T \frac{2g(r)}{\alpha_n} + \int_{t_0}^{t_1} \int_{B_{R/2}(0)} \tau \|b_h\|_\infty\, dx\, dt \leqq T \frac{2g(r)}{\alpha_n} + c_n T R^n \tau \|b_h\|_\infty.
\end{aligned}
$$

The thesis follows applying the Riesz-Fréchet-Kolmogorov compactness criterion (see Lemma C.2), recalling that $\{b_h\}$ is uniformly bounded in $L^\infty([0, T] \times \mathbb{R}^n)$. $\square$

Corollary 4.3 (Existence of the flow). *Let b be a bounded vector field satisfying assumption* (R3) *and such that* $[\operatorname{div} b]^- \in L^1\big([0, T]; L^\infty(\mathbb{R}^n)\big)$. *Then there exists a regular Lagrangian flow associated to b.*

Proof. It is sufficient to regularize b with a positive convolution kernel in $\mathbb{R}^n$ and apply Corollary 4.2. It is simple to check that the regularized vector fields satisfy the equi-bounds needed for the compactness result, due to the convexity of the map $z \mapsto z \log(2+z)$ for $z \geqq 0$. $\square$

5. Lipexp$_p$-regularity for transport equations with $W^{1,p}$ coefficients

In this section we show that solutions to transport equations with Sobolev coefficients propagate a very mild regularity property of the initial data.

Definition 5.1 (The space Lipexp$_p$). We say that a function $f : E \subset\subset \mathbb{R}^n \to \mathbb{R}^k$ belongs to Lipexp$_p(E)$ if for every $\varepsilon > 0$ there exists a measurable set $K \subset E$ such that

(i) $|E \backslash K| \leqq \varepsilon$;

(ii) $\mathrm{Lip}(f|_K) \leqq \exp(C\varepsilon^{-1/p})$ for some constant $C < \infty$ independent on ε.

Moreover we denote by $|f|_{\mathrm{LE}_p(E)}$ the smallest constant C such that the conditions above hold.

Remark 5.2. Note that:

• Lipexp$_\infty$ is the space of functions which coincide with a Lipschitz function almost everywhere.

• $|f|_{\mathrm{LE}_p(E)}$ is not homogeneous, and then it is not a norm, and can be explicitely defined as

$$|f|_{\mathrm{LE}_p(E)} := \sup_{\varepsilon>0}\{\varepsilon^{1/p} \log \min\{\mathrm{Lip}(f|_K) : |E\backslash K| \leqq \varepsilon\}\}.$$

• One can compare this definition with a similar result for Sobolev functions: if $f \in W^{1,p}(E;\mathbb{R}^k)$, then for every $\varepsilon > 0$ there exists a set $K \subset E$ such that $|E\backslash K| \leqq \varepsilon$ and $\mathrm{Lip}(f|_K) \leqq \|Df\|_{L^p(E)}\varepsilon^{-1/p}$.

Theorem 5.3. *Let b be a vector field satisfying assumptions* (R1) *and* (R2) *and such that* $\operatorname{div} b \in L^1\big([0,T];L^\infty(\mathbb{R}^n)\big)$. *Let $\bar u \in L^\infty(\mathbb{R}^n)$ such that $\bar u \in$ Lipexp$_p(\Omega)$ for every $\Omega \subset\subset \mathbb{R}^n$. Let u be the solution of the Cauchy problem*

$$\begin{cases} \partial_t u(t,x) + b(t,x)\cdot\nabla_x u(t,x) = 0, \\ u(0,\cdot) = \bar u. \end{cases} \tag{28}$$

Then for every $\Omega \subset\subset \mathbb{R}^n$ we have that

$$\sup_{0\leqq t\leqq T} |u(t,\cdot)|_{\mathrm{LE}_p(\Omega)} < \infty.$$

Remark 5.4. Since $u \in C\big([0,T], L^1_{\mathrm{loc}}(\mathbb{R}^n) - w\big)$, we can define $u(t,\cdot)$ for every $t \in [0,T]$.

Proof of Theorem 5.3. Let X be the regular Lagrangian flow generated by b. Then:

(a) There exists a constant $C > 0$ such that $C^{-1}|\Omega| \leqq |X(t,\Omega)| \leqq C|\Omega|$ for every $t \in [0,T]$ and for every $\Omega \subset \mathbb{R}^n$; therefore, for every $t \in [0,T]$, we can define $\Psi(t,x)$ via the identity $X\big(t,\Psi(t,x)\big) = \Psi\big(t,X(t,x)\big) = x$ for a.e. $x \in \mathbb{R}^n$.

(b) For every t we have $u(t,x) = \bar{u}\big(\Psi(t,x)\big)$ for almost every x.

Note that if for every t we consider the regular Lagrangian flow $\Phi(t,\cdot,\cdot)$ of

$$\begin{cases} \dfrac{d\Phi}{d\tau}(t,\tau,x) = -b\big(t-\tau,\Phi(t,\tau,x)\big), \\ \Phi(t,0,x) = x, \end{cases}$$

then $\Psi(t,x) = \Phi(t,t,x)$. Therefore, thanks to Proposition 3.4 we conclude that

$$\sup_{0 \leqq t \leqq T} |\Psi(t,\cdot)|_{\mathrm{LE}_p(\Omega)} \leqq C_1(\Omega)$$

for every $\Omega \subset\subset \mathbb{R}^n$.

Let $t \in [0,T]$, $R > 0$ and $\varepsilon > 0$ be given. Choose $K_1 \subset B_R(0)$ such that

- $|B_R(0) \backslash K_1| \leqq \varepsilon/3$;
- $\mathrm{Lip}\big(\Psi(t,\cdot)|_{K_1}\big) \leqq \exp\big(|\Psi(t,\cdot)|_{\mathrm{LE}_p(B_R(0))}(\varepsilon/3)^{-1/p}\big)$.

Applying Proposition 3.2 we can find $\bar{R} > 0$ such that

$$\big|\Psi\big(t,B_R(0)\big) \backslash B_{\bar{R}}(0)\big| \leqq \frac{\varepsilon}{3C},$$

where C is the constant in (a). Now, select $K_2 \subset B_{\bar{R}}(0)$ such that

- $|B_{\bar{R}}(0) \backslash K_2| \leqq \varepsilon/3C$;
- $\mathrm{Lip}(\bar{u}|_{K_2}) \leqq \exp\big(|\bar{u}|_{\mathrm{LE}_p(B_{C(R)}(0))}(\varepsilon/3C)^{-1/p}\big)$,

where again C is as in (a). Next consider $K := K_1 \cap \big(\Psi(t,\cdot)\big)^{-1}(K_2) = K_1 \cap X(t,K_2)$. Since

$$\begin{aligned} B_R(0) \backslash K &\subset \big(B_R(0) \backslash K_1\big) \cup \big(B_R(0) \backslash X(t,K_2)\big) \\ &\subset \big(B_R(0) \backslash K_1\big) \cup X\big(t,\Psi\big(t,B_R(0)\big) \backslash B_{\bar{R}}(0)\big) \cup X\big(t,B_{\bar{R}}(0) \backslash K_2\big), \end{aligned}$$

we have

$$|B_R(0) \backslash K| \leqq |B_R(0) \backslash K_1| + \big|X\big(t,\Psi\big(t,B_R(0)\big) \backslash B_{\bar{R}}(0)\big)\big| + \big|X\big(t,B_{\bar{R}}(0) \backslash K_2\big)\big| \leqq \varepsilon.$$

Given $x, y \in K$ we have $\Psi(t,x), \Psi(t,y) \in K_2$ and hence we can estimate

$$
\begin{aligned}
|u(t,x)-u(t,y)| &= \big|\bar{u}\big(\Psi(t,x)\big)-\bar{u}\big(\Psi(t,y)\big)\big| \leqq \mathrm{Lip}(\bar{u}|_{K_2})|\Psi(t,x)-\Psi(t,y)| \\
&\leqq \mathrm{Lip}(\bar{u}|_{K_2})\,\mathrm{Lip}\big(\Psi(t,\cdot)|_{K_1}\big)|x-y| \\
&= |x-y|\exp\{[(3C)^{1/p}|\bar{u}|_{\mathrm{LE}_p(B_{\bar{R}}(0))}+3^{1/p}|\Psi(t,\cdot)|_{\mathrm{LE}_p(B_R(0))}]\varepsilon^{-1/p}\}.
\end{aligned}
$$

Therefore $\varepsilon^{1/p}\log\big(\mathrm{Lip}\big(u(t,\cdot)|_K\big)\big)$ is bounded by a constant independent of ε and t (but which depends on R). Taking the supremum over t and ε, we conclude that

$$
\sup_{0\leqq t\leqq T}|u(t,\cdot)|_{\mathrm{LE}_p(B_R(0))}\leqq C(R),
$$

and this conclude the proof. $\square$

6. An application to a conjecture on mixing flows

In [7] the author considers a problem on mixing vector fields on the two-dimensional torus $K=\mathbb{R}^2/\mathbb{Z}^2$. In this section, we are going to show that the Lipschitz estimate of Proposition 3.4 gives an answer to this problem, although in the L^p setting $(p>1)$ instead of the L^1 setting considered in [7].

Fix coordinates $x=(x_1,x_2)\in[0,1[\times[0,1[$ on K and consider the set

$$
A=\{(x_1,x_2):0\leqq x_2\leqq 1/2\}\subset K.
$$

If $b:[0,1]\times K\to\mathbb{R}^2$ is a smooth time-dependent vector field, we denote as usual by $X(t,x)$ the flow of b and by $\Phi:K\to K$ the value of the flow at time $t=1$. We assume that the flow is nearly incompressibile, so that for some $\kappa'>0$ we have

$$
\kappa'|\Omega|\leqq|X(t,\Omega)|\leqq\frac{1}{\kappa'}|\Omega| \tag{29}
$$

for all $\Omega\subset K$ and all $t\in[0,1]$. For a fixed $0<\kappa<1/2$, we say that Φ *mixes the set A up to scale ε* if for every ball $B_\varepsilon(x)$ we have

$$
\kappa|B_\varepsilon(x)|\leqq|B_\varepsilon(x)\cap\Phi(A)|\leqq(1-\kappa)|B_\varepsilon(x)|.
$$

Then in [7] the following conjecture is proposed:

Conjecture 6.1 (Bressan's mixing conjecture). *Under these assumptions, there exists a constant C depending only on κ and κ' such that, if Φ mixes the set A up to scale ε, then*

$$
\int_0^1\int_K|D_xb|\,dx\,dt\geqq C|\log\varepsilon|\quad \textit{for every } 0<\varepsilon<1/4.
$$

In this section, we show the following result:

Theorem 6.2. *Let $p > 1$. Under the previous assumptions, there exists a constant C depending only on κ, κ' and p such that, if Φ mixes the set A up to scale ε, then*

$$\int_0^1 \|D_x b\|_{L^p(K)}\,dt \geqq C|\log \varepsilon| \quad \textit{for every } 0 < \varepsilon < 1/4.$$

Proof. We set $M = \|D_x b\|_{L^1([0,1];L^p(K))}$ and $A' = K \backslash A$. Applying Proposition 3.4, and noticing that the flow is bounded since we are on the torus, for every constant $\eta > 0$ we can find a set B with $|B| \leqq \eta$ such that

$$\mathrm{Lip}(\Phi^{-1}|_{K\backslash B}) \leqq \exp(\beta M), \tag{30}$$

where the constant β depends only on κ', η and p. Since Φ mixes the set A up to scale ε, for every $x \in A$ we have

$$\big|B_\varepsilon\big(\Phi(x)\big) \cap \Phi(A')\big| \geqq \kappa\big|B_\varepsilon\big(\Phi(x)\big)\big|. \tag{31}$$

We define

$$\tilde{A} = \big\{x \in A : B_\varepsilon\big(\Phi(x)\big) \cap [\Phi(A')\backslash B] = \emptyset\big\}.$$

From this definition and from (31) we get that for every $x \in \tilde{A}$

$$\big|B_\varepsilon\big(\Phi(x)\big) \cap B\big| \geqq \kappa\big|B_\varepsilon\big(\Phi(x)\big)\big|. \tag{32}$$

From (32) and the Besicovitch covering theorem we deduce that for an absolute constant c we have

$$|\Phi(\tilde{A})| \leqq \frac{c}{\kappa}|B| \leqq \frac{c\eta}{\kappa}.$$

From the compressibility condition (29) we deduce

$$|\tilde{A}| \leqq \frac{c\eta}{\kappa\kappa'}.$$

Since, using again (29), we know that

$$|\Phi^{-1}(B)| \leqq \frac{|B|}{\kappa'} \leqq \frac{\eta}{\kappa'},$$

we can choose $\eta > 0$, depending on κ and κ' only, in such a way that

$$|\tilde{A}| + |\Phi^{-1}(B)| \leqq \frac{1}{6}.$$

This implies the existence of a point $\bar{x} \in A\backslash[\tilde{A} \cup \Phi^{-1}(B)]$ with $\mathrm{dist}(\bar{x}, A') \geqq 1/6$. Let $\bar{y} = \Phi(\bar{x})$. Since $\bar{x} \notin \tilde{A}$, we can find a point $\bar{z} \in B_\varepsilon(\bar{y}) \cap [\Phi(A')\backslash B]$. Clearly we have $|\bar{y} - \bar{z}| \leqq \varepsilon$ and (since $\Phi^{-1}(z) \in A'$) we also have $|\bar{x} - \Phi^{-1}(\bar{z})| \geqq 1/6$.

Since $\bar{y}, \bar{z} \notin B$, we can apply (30) to deduce

$$\frac{1}{6} \leqq \varepsilon \operatorname{Lip}(\Phi^{-1}|_{K\setminus B}) \leqq \varepsilon \exp(\beta M),$$

where now β depends only on κ, κ' and p, since η has been fixed. This implies that

$$M = \|D_x b\|_{L^1([0,1];L^p(K))} \geqq \frac{1}{\beta} \log\left(\frac{1}{6\varepsilon}\right).$$

Hence we can find $\varepsilon_0 > 0$ such that

$$M \geqq \frac{1}{2\beta} |\log \varepsilon| \quad \text{for every } 0 < \varepsilon < \varepsilon_0.$$

We are now going to show the thesis for every $0 < \varepsilon < 1/4$. Indeed, suppose that the thesis is false. Then, we could find a sequence $\{b_h\}$ of vector fields and a sequence $\{\varepsilon_h\}$ with $\varepsilon_0 < \varepsilon_h < 1/4$ in such a way that

$$\|D_x b_h\|_{L^1([0,1];L^p(K))} \leqq \frac{1}{h} |\log \varepsilon_h|$$

and the corresponding map Φ_h mixes the set A up to scale ε_h. This implies that

$$\|D_x b_h\|_{L^1([0,1];L^p(K))} \leqq \frac{1}{h} |\log \varepsilon_h| \leqq \frac{1}{h} |\log \varepsilon_0| \to 0 \quad \text{as } h \to \infty.$$

Up to an extraction of a subsequence, we can assume that $\varepsilon_h \to \bar{\varepsilon}$ and that $\Phi_h \to \Phi$ strongly in $L^1(K)$. For this, we apply the compactness result in Theorem 3.6, noticing that (29) gives a uniform control on the compressibility constants of the flows and that we do not need any assumption on the growth of the vector fields, since we are on the torus and then the flow is automatically uniformly bounded. Now, notice that the mixing property is stable with respect to strong convergence: this means that Φ has to mix up to scale $\bar{\varepsilon} \leqq 1/4$. But since $\|D_x b_h\|_{L^1([0,1];L^p(K))} \to 0$, we deduce that Φ is indeed a translation on K, hence it cannot mix the set A up to a scale which is smaller than $1/4$. From this contradiction we get the thesis. $\square$

Remark 6.3. We notice that the constant $1/4$ in Theorem 6.2 depends on the shape of the set A: this bound comes from the fact that a translation does not mix up to a scale $\varepsilon < 1/4$. Our proof can be easily extended to the case of a measurable set A with any shape, giving a different upper bound for the values of ε such that the result is true.

Appendix A. Maximal functions

In this first appendix, we recall the definition of the *local maximal function* of a locally finite measure and of a locally summable function and we recollect some well-known properties which are used throughout all this paper.

Definition A.1 (Local maximal function). Let μ be a (vector-valued) locally finite measure. For every $\lambda > 0$, we define the *local maximal function* of μ as

$$M_\lambda \mu(x) = \sup_{0<r<\lambda} \frac{|\mu|\big(B_r(x)\big)}{|B_r(x)|} = \sup_{0<r<\lambda} \fint_{B_r(x)} d|\mu|(y), \quad x \in \mathbb{R}^n.$$

When $\mu = f\mathscr{L}^n$, where f is a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$, we will often use the notation $M_\lambda f$ for $M_\lambda \mu$.

The proof of the following two lemmas can be found in [12].

Lemma A.2. *Let $\lambda > 0$. The local maximal function of μ is finite for a.e. $x \in \mathbb{R}^n$ and we have*

$$\int_{B_\rho(0)} M_\lambda f(y)\, dy \leqq c_{n,\rho} + c_n \int_{B_{\rho+\lambda}(0)} |f(y)| \log\big(2 + |f(y)|\big)\, dy.$$

For $p > 1$ and $\rho > 0$ we have

$$\int_{B_\rho(0)} \big(M_\lambda f(y)\big)^p\, dy \leqq c_{n,p} \int_{B_{\rho+\lambda}(0)} |f(y)|^p\, dy,$$

but this is false for $p = 1$. For $p = 1$ we have the weak estimate

$$|\{y \in B_\rho(0) : M_\lambda f(y) > \alpha\}| \leqq \frac{c_n}{\alpha} \int_{B_{\rho+\lambda}(0)} |f(y)|\, dy, \tag{33}$$

for every $\alpha > 0$.

Lemma A.3. *If $u \in BV(\mathbb{R}^n)$ then there exists a negligible set $N \subset \mathbb{R}^n$ such that*

$$|u(x) - u(y)| \leqq c_n |x - y| \big(M_\lambda Du(x) + M_\lambda Du(y)\big)$$

for $x, y \in \mathbb{R}^n \setminus N$ with $|x - y| \leqq \lambda$.

We also recall the Chebyshev inequality:

$$|\{|f| > t\}| \leqq \frac{1}{t} \int_{\{|f|>t\}} |f(x)|\, dx \leqq \frac{|\{|f| > t\}|^{1/q}}{t} \|f\|_{L^p(\Omega)},$$

which implies

$$|\{|f| > t\}|^{1/p} \leqq \frac{\|f\|_{L^p(\Omega)}}{t}. \tag{34}$$

Appendix B. Convergence in measure and approximate differentiability

We recall that a sequence of Borel maps $\{f_h\}$ is said to be *locally convergent in measure* to f if

$$\lim_{h\to\infty} |\{x \in B_R(0) : |f_h(x) - f(x)| > \delta\}| = 0 \quad \text{for every } R > 0 \text{ and } \delta > 0.$$

This convergence is equivalent to the fact that

$$1 \wedge |f_h - f| \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

If the sequence $\{f_h\}$ is locally equi-bounded in L^∞, then the local convergence in measure is equivalent to the strong convergence in L^1_{loc}.

We say that a Borel map $f : \mathbb{R}^n \to \mathbb{R}^k$ is *approximately differentiable* at $x \in \mathbb{R}^n$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^k$ such that the difference quotients

$$y \mapsto \frac{f(x + \varepsilon y) - f(x)}{\varepsilon}$$

locally converge in measure as $\varepsilon \downarrow 0$ to Ly. This is clearly a local property. Equivalently, the approximative differentiability condition can be stated in the following way: there exists a map $\tilde{f}$, differentiable in the classical sense at x, such that $\tilde{f}(x) = f(x)$ and the coincidence set $\{y : \tilde{f}(y) = f(y)\}$ has density 1 at x. This characterization, together with Rademacher theorem and some extension arguments, shows that if $f|_K$ is a Lipschitz map for some set $K \subset \mathbb{R}^n$, then f is approximately differentiable at almost every point of K. In the following theorem we show a kind of converse of this statement: an approximately differentiable map can be approximated, in the Lusin sense, with Lipschitz maps.

Theorem B.1. *Let $f : \Omega \to \mathbb{R}^k$. Assume that there exist A_h such that $\left|\Omega \setminus \bigcup_h A_h\right| = 0$ and $f|_{A_h}$ is Lipschitz for any h. Then f is approximately differentiable at a.e. $x \in \Omega$. Conversely, if f is approximately differentiable at all points of $\Omega' \subset \Omega$, we can write Ω' as a countable union of sets A_h such that $f|_{A_h}$ is Lipschitz for any h (up to a redefinition on a negligible set).*

For the proof, see [10], Theorem 3.1.16.

Appendix C. Compactness

In this appendix we give some "abstract" results which have been used in the previous sections to prove compactness for the regular Lagrangian flows.

Lemma C.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Borel set and let $\{f_h\}$ be a sequence of maps into $\mathbb{R}^m$. Suppose that for every $\delta > 0$ we can find a positive constant $M_\delta < \infty$ and, for every fixed h, a Borel set $B_{h,\delta} \subset \Omega$ with $|\Omega \setminus B_{h,\delta}| \leqq \delta$ in such a way that*

$$\|f_h\|_{L^\infty(B_{h,\delta})} \leqq M_\delta$$

and

$$\mathrm{Lip}(f_h|_{B_{h,\delta}}) \leqq M_\delta.$$

Then the sequence $\{f_h\}$ is precompact in measure in Ω.

Proof. For every $j \in \mathbb{N}$ we find the value $M_{1/j}$ and the sets $B_{h,1/j}$ as in the assumption of the lemma, with $\delta = 1/j$. Now, arguing component by component, we can extend every map $f_h|_{B_{h,1/j}}$ to a map f_h^j defined on Ω in such a way that the equi-bounds are preserved, up to a dimensional constant: we have

$$\|f_h^j\|_{L^\infty(\Omega)} \leqq M_{1/j} \quad \text{for every } h$$

and

$$\mathrm{Lip}(f_h^j) \leqq c_n M_{1/j} \quad \text{for every } h.$$

Then we apply the Ascoli-Arzelà theorem (notice that by uniform continuity all the maps f_h^j can be extended to the compact set $\overline{\Omega}$) and using a diagonal procedure we find a subsequence (in h) such that for every j the sequence $\{f_h^j\}_h$ converges uniformly in Ω to a map f_∞^j.

Now we fix $\varepsilon > 0$. We choose $j \geqq 3/\varepsilon$ and we find $N = N(j)$ such that

$$\int_\Omega |f_i^j - f_k^j|\, dx \leqq \varepsilon/3 \quad \text{for every } i, k > N.$$

Keeping j and $N(j)$ fixed we estimate, for $i, k > N$,

$$\begin{aligned} \int_\Omega 1 \wedge |f_i - f_k|\, dx &\leqq \int_\Omega 1 \wedge |f_i - f_i^j|\, dx + \int_\Omega 1 \wedge |f_i^j - f_k^j|\, dx + \int_\Omega 1 \wedge |f_k^j - f_k|\, dx \\ &\leqq |\Omega \backslash B_{i,1/j}| + \int_\Omega |f_i^j - f_k^j|\, dx + |\Omega \backslash B_{k,1/j}| \\ &\leqq \frac{1}{j} + \frac{\varepsilon}{3} + \frac{1}{j} \leqq \varepsilon. \end{aligned}$$

It follows that the given sequence has a subsequence which is Cauchy with respect to the convergence in measure in Ω. This implies the thesis. □

We also recall the following classical criterion for strong compactness in L^p, since we used it during the proof of Corollary 4.2.

Lemma C.2 (Riesz-Fréchet-Kolmogorov compactness criterion). *Let $\mathscr{F}$ be a bounded subset of $L^p(\mathbb{R}^N)$ for some $1 \leqq p < \infty$. Suppose that*

$$\lim_{|h| \to 0} \|f(\cdot - h) - f\|_p = 0 \quad \textit{uniformly in } f \in \mathscr{F}.$$

Then $\mathscr{F}$ is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$.

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